Lecture 1.

I Group constructions $\boxed{\text{Amalgamated products and HNN extensions}}$

(IA.) Consider G_1, G_2 two (~~finitely presented~~) groups and $A \xrightarrow{f_i} G_i$ morphisms

$\boxed{\text{Theorem 1}}$: There exists an unique group denoted $G_1 \underset{A}{*} G_2$ endowed with homomorphisms

$$\begin{tikzcd} & G_1 \arrow[dr, "h_1"] & \\ A \arrow[ur, "f_1"] \arrow[rr, "f"] \arrow[dr, "f_2"'] & & G_1 \underset{A}{*} G_2 \\ & G_2 \arrow[ur, "h_2"'] & \end{tikzcd}$$

which is universal i.e. for any commutative diagram

$$\begin{tikzcd} & G_1 \arrow[dr, "g_1"] & \\ A \arrow[ur, "f_1"] \arrow[rr, "g"] \arrow[dr, "f_2"'] & & G \\ & G_2 \arrow[ur, "g_2"'] & \end{tikzcd}$$

then exists a unique $h: G_1 \underset{A}{*} G_2 \to G$ making the diagram

$$\begin{tikzcd} & G_1 \arrow[d, "g_1"] \arrow[dr, "h_1"] & \\ A \arrow[ur, "f_1"] \arrow[r, "g"] \arrow[dr, "f_2"'] & G & G_1 \underset{A}{*} G_2 \arrow[l, "h"'] \\ & G_2 \arrow[u] \arrow[ur, "h_2"'] & \end{tikzcd}$$

commutative. We call $G_1 \underset{A}{*} G_2$ amalgamated product of G_1, G_2 over A.

Proof Consider presentations $G_1 = \langle x_i \mid r_j \rangle$, $G_2 = \langle y_i \mid s_j \rangle$ and $\{a_i\}$ system of generators for A. Let us set

$$G_1 \underset{A}{*} G_2 := \langle x_i, y_j \mid r_j, s_k, f_1(a_m) f_2(a_m)^{-1}, \text{ all } i, j, k, m \rangle$$

It is immediate that $G_1 \underset{A}{*} G_2$ satisfies the universality property above. The uniqueness of the universal object $G_1 \underset{A}{*} G_2$ follows: if H is any other universal group as above then we have maps $G_1 \underset{A}{*} G_2 \underset{h'}{\overset{h}{\rightleftarrows}} H$ making

$$\begin{tikzcd} & G_1 \arrow[dl, "h_1'"'] \arrow[dr, "h_1"] & \\ H \arrow[rr, shift left, "h'"] & & G_1 \underset{A}{*} G_2 \arrow[ll, shift left, "h"] \\ & G_2 \arrow[ul, "h_2'"] \arrow[ur, "h_2"'] & \end{tikzcd}$$

and thus

$$\begin{tikzcd} & G_1 \arrow[dl, "h_1"'] \arrow[dr, "h_1"] & \\ G_1 \underset{A}{*} G_2 \arrow[rr, "h' \circ h"] & & G_1 *_A G_2 \\ & G_2 \arrow[ul, "h_2"] \arrow[ur, "h_2"'] & \end{tikzcd}$$

; uniqueness of the map $h' \circ h$ implies $h' \circ h = id$ and similar $h \circ h' = id$. $\square$

$\boxed{\text{Exercice}}$ 1) $\mathbb{Z} \underset{\{1\}}{*} \mathbb{Z} = \mathbb{F}(2)$; if $A = \{1\}$ call it free product

2) $\mathbb{Z}/p\mathbb{Z} \underset{\mathbb{Z}}{*} \mathbb{Z}/q\mathbb{Z} = \{1\}$ if $\gcd(p, q) = 1$.

(I B) Generalization. Graph of groups and inductive limits.

Construction 1: Given $\{G_i\}_{i\in I}$ family of groups and for each $(i,j)\in I\times I$ a family F_{ij} of homomorphisms $f: G_i \longrightarrow G_j$ (F_{ij} might be empty).

Define $H = \varinjlim G_i$ to be a group H endowed with morphisms $g_i : G_i \to H$ such that $\forall f \in F_{ij}$ the diagram below commutes:

$$\begin{tikzcd} G_i \arrow[d, "F_{ij}\ni f"'] \arrow[dr, "g_i"] & \\ G_j \arrow[r, "g_j"'] & H \end{tikzcd}$$

with the universal property i.e. for any G group, $g_i' : G_i \to G$ morphisms such that $\begin{tikzcd} G_i \arrow[d, "F_{ij}\ni f"'] \arrow[dr, "g_i'"] & \\ G_j \arrow[r, "g_j'"'] & G \end{tikzcd}$ commutes then exists a unique homomorphism $H \longrightarrow G$ making

$$\begin{tikzcd} & G_i \arrow[dl, "g_i'"'] \arrow[dr, "h_i"] & \\ G & & H \arrow[ll, "h"] \end{tikzcd}$$

commutative $\forall i$.

$\boxed{\text{Theorem 2}}$ $\boxed{\varinjlim G_i \text{ exists and it is unique.}}$

Proof: as above in Th. 1. $\square$

$\boxed{\text{Example}}$: Taking $\{A \xrightarrow{f_i} G_i\}_i$ one obtains $\underset{A}{*} G_i$ the iterated amalgamated product of G_i over A. This operation is associative.

Construction 2. A graph of groups (over the graph Γ) is the assignment of a group G_v for any vertex v of Γ and a group G_e for any edge e of Γ together with homomorphisms (in general injective)

$$v \overset{e}{\circ\!\!-\!\!\!-\!\!\!-\!\!\circ} w \qquad G_v \xleftarrow{\varphi_{v,e}} G_e \xrightarrow{\varphi_{w,e}} G_w$$

for any edge e with endpoints v, w. Let $E(\Gamma), V(\Gamma)$ the edges, vertices. The fundamental group $\mathcal{G}$ of the graph of groups is the group satisfying the universality property from Construction 1. However here is a direct definition; take $T \subset \Gamma$ be a spanning tree and choose a generator y_e for each oriented edge $e \in E(\Gamma) - T$, ~~along with an~~

$$\mathcal{G} = \left\langle G_x, x \in V(\Gamma),\ y_e, e\in E(\Gamma)-T \;\middle|\; \begin{array}{l} y_{\bar e} = y_e^{-1}, \text{ where } \bar e = e \text{ with opposite orient} \\ y_e \varphi_{v,e}(x)\, y_e^{-1} = \varphi_{w,e}(x) \ \forall e \in E(\Gamma) \end{array}\right.$$

with $y_e = 1$ if $e \in T$

Example $\boxed{1)}$ $\Gamma = \overset{G_1}{\circ}\underset{A}{\text{———}}\overset{G_2}{\circ}$ get $\mathcal{G} = G_1 \underset{A}{*} G_2$

$\boxed{2)}$ $\Gamma = G \circ\!\!\bigcirc A$ correspond to having two maps

$f_1 : A \to G$, $f_2 : A \to G$. Then $\mathcal{G} = \langle G ; y \mid y f_1(a) y^{-1} = f_2(a) \ \forall a \in A \rangle$

it is called the HNN extension.

Remark: If $\varphi_{v,e}$ are all monomorphisms then $G_v, G_e \hookrightarrow \mathcal{G}$.
Moreover the cosets $\mathcal{G}/G_v$, $\mathcal{G}/G_e$ are the vertices and the edges sets of a graph Y (called the universal covering tree) such that $\mathcal{G}$ acts on Y with $Y/\mathcal{G} \simeq \Gamma$. The stabilizers of edges and vertices are the groups G_e, G_v ...

(I C) Interpretation topologique : Van Kampen Theorem:

Let X be a topological space and $U_1, U_2 \subset X$ be open path-connected subspaces for which $U_1 \cap U_2$ is open path-connected, $x \in U_1 \cap U_2$.
Then

$$\pi_1(U_1 \cup U_2, x) \simeq \pi_1(U_1, x) \underset{\pi_1(U_1 \cap U_2, x)}{*} \pi_2(U_2, x)$$

where $f_i : \pi_1(U_1 \cap U_2, x) \longrightarrow \pi_1(U_i, x)$ is the map induced in homotopy by the inclusion $U_1 \cap U_2 \hookrightarrow U_i$.

Rk: This is still valid if $U_i, U_1 \cap U_2$ are closed path-connected subspaces.

Consider now X a topological space and $A, B \subset X$ two closed subspaces with a homeomorphism $\theta : A \xrightarrow{\sim} B$. Assume that the inclusion maps $\pi_1(A) \xrightarrow{h} \pi_1(X)$, $\pi_1(B) \xrightarrow{h'} \pi_1(X)$ are injective.
Let then Y be the space

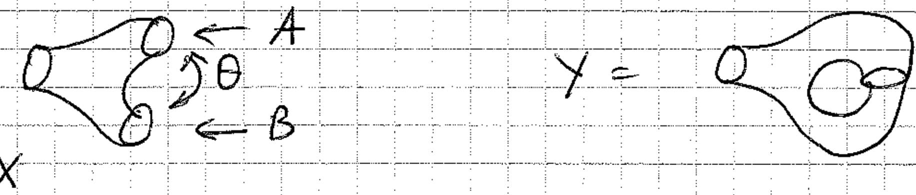
$$Y = X / \{a \sim \theta(a) \text{ for } a \in A\}$$

Then

$$\pi_1(Y) \simeq \langle \pi_1(X), t \mid t a t^{-1} = \theta(a), \ \forall a \in \pi_1 A \rangle$$

where $\theta : h(\pi_1 A) \longrightarrow h(\pi_1 B.)$ ~~identifie~~ is the map induced by θ.
This is a HNN extension.

Exemple



(I D) Van der Waerden structure theorem.

Let $\begin{array}{l} A \longrightarrow G_1 \\ \quad\searrow G_2 \end{array}$ be injective morphisms. Let S_i set of cosets G_i/A s.t. $1 \in S_i$. Consider $(j_1, \dots, j_n)$, $n \geq 0$, $j_k \in \{1,2\}$ with $j_k \neq j_{k+1}$ $\forall k$ i.e. alternate sequence. Let $A \xrightarrow{f} G_1 \ast_A G_2$, $G_i \xrightarrow{f_i} G_1 \ast_A G_2$.

Definition The word $a s_1 s_2 \dots s_n$ with $a \in A$, $s_j \in S_j$ and $s_j \neq 1$ is called a reduced word. The sequence $(j_1, \dots, j_n)$ is part of the data.

Theorem 3: Let $g \in G_1 \ast_A G_2$, $g \neq 1$. Then there exists a unique reduced word $a s_1 \dots s_n$ such that

$$(\ast) \qquad g = f(a)\, f_1(s_1)\, f_2(s_2) \dots f_{j_n}(s_n)$$

Proof 1) $G_1 \cup G_2$ generates $G_1 \ast_A G_2$ and thus $g = f_{j_1}(g_1) \dots f_{j_n}(g_n)$, with $g_i \in G_{j_i}$. We write then $g_n = a_n s_n$, $a_n \in A$, $s_n \in S_{j_n}$. Then

$$g = \dots f_{j_n}(a_n s_n) = \dots f_{j_{n-1}}(g_{n-1})\, f_{j_n}(a_n)\, f_{j_n}(s_n) = \dots f_{j_{n-1}}(g_{n-1} a_n)\, f_{j_n}(s_n)$$

Now $g_{n-1} a_n \in G_{j_{n-1}}$ and it can be written as $g_{n-1} a_n = a_{n-1} s_{n-1}$, with $a_{n-1} \in A$, $s_{n-1} \in S_{j_{n-1}}$. We continue in this way to push the a's towards the begining of the word and get $(\ast)$.

2) Let us prove the uniqueness of the reduced word. Let $\mathcal{X}$ be the set of all reduced words. We define an action (to the left):

$$G_1 \ast_A G_2 \times \mathcal{X} \longrightarrow \mathcal{X}$$

by using the universality of $G_1 \ast_A G_2$; thus we will define actions of G_1 and G_2 such that their restriction to the images of A coincide. Specifically consider $\mathcal{X} = D_1^{(i)} \cup D_2^{(i)}$ where

$$D_1^{(i)} = \{ a s_1 s_2 \dots s_n \,,\; s_1 \in S_{j_1} - \{1\} \text{ and } j_1 \neq i \}$$

and

$$D_2^{(i)} = \{ a s_1 s_2 \dots s_n \, ; \ s_1 \in S_i - 1 \}.$$

For fixed $i \in \{1,2\}$ then $D_1^{(i)}, D_2^{(i)}$ is a partition of $\mathcal{X}$.
Let now define the action

$$G_i \times \mathcal{X} \longrightarrow \mathcal{X}$$

by means of

$$G_i \times D_1^{(i)} \longrightarrow D_1^{(i)}, \quad G_i \times D_2^{(i)} \longrightarrow D_2^{(i)}$$

• $(g, a s_1 s_2 \dots s_n) \longrightarrow (g a) s_1 s_2 \dots s_n = (a' s_0) s_1 s_2 \dots s_n$
where we choose $s_0 \in S_{j_0}$ such that

$$g a = a' s_0 \quad , \ a' \in A$$

Thus a' is uniquely determined because $A \hookrightarrow G_i$.

•• $(g, a s_1 \dots s_n) \longrightarrow (g a s_1) s_2 \dots s_n = (a' s_1') s_2 \dots s_n$

where we choose $s_1' \in S_{j_0}$ and $a' \in A$ such that

$$g a s_1 = a' s_1'.$$

Observe that G_i is indeed a left action on $\mathcal{X}$ i.e

$$g_1 (g_2 \cdot x) = g_1 g_2 \cdot x \qquad \forall \ g_1, g_2 \in G_i, \ x \in \mathcal{X}.$$

The restrictions of the two actions at $A \subset G_1$ and at $A \subset G_2$ coincide because

$$a_1 (a_2 s_1 s_2 \dots s_n) = (a_1 a_2) s_1 s_2 \dots s_n.$$

The universality property implies that we have an action

$$G_1 \underset{A}{*} G_2 \times \mathcal{X} \longrightarrow \mathcal{X} \ , \ (g, x) \longrightarrow g \cdot x.$$

Let $E : X \to G_1 \underset{A}{*} G_2$ be the evaluation map

$$E(a s_1 s_2 \dots s_n) = f(a) f_{j_1}(s_1) \dots f_{j_n}(s_n) \in G_1 \underset{A}{*} G_2.$$

We claim that the action of $E(w)$ on the word trivial $\mathbb{1} \in \mathcal{X}$ is tautological i.e.

$(**) \qquad E(x) \cdot \mathbb{1} = x$

Indeed

$$f(a) f_{j_1}(s_1) \dots f_{j_n}(s_n) \cdot \mathbb{1} = f(a) \dots f_{j_{n-1}}(s_{n-1}) \cdot s_n =$$

$$= f(a) \dots f_{j_{n-2}}(s_{n-2}) \cdot s_{n-1} s_n = \dots = a s_1 \dots s_n$$

Therefore if $E(x) = E(x')$ then $x = x' \in \mathcal{X}$ and so uniqueness ok $\square$

I E Britton's lemma.

This is an analogue of the Van der Waerden structure theorem, for HNN extensions.

Theorem 4: Let $H, K \subset G$, $\theta : H \to K$ isomorphism, $G = \langle S | R \rangle$

$$G *_\theta = \langle S, t \mid R, \; tht^{-1} = \theta(h), \; \forall h \in H \rangle$$

the HNN extension. Let w be a word of the form

$$w = g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots g_{n-1} t^{\varepsilon_n} g_n$$

$$g_i \in G, \quad \varepsilon_i \in \{-1, 1\}.$$

Assume that

- either $n = 0$ and $g_0 \neq 1$ in G.
- or $n > 0$ and there is no subwords of the form

$$t g_j t^{-1}, \; g_j \in H$$

$$t^{-1} g_k t, \; g_k \in K.$$

Then $w \neq 1$ in $G *_\theta$.

Most basic properties result from this, as follows:

Consequences

i) $G \hookrightarrow G *_\theta$ is injective

ii) if $H \neq G$, $K \neq G$ then $G *_\theta \supset$ subgroup $\mathbb{F}(2)$.

iii) $A \hookrightarrow G_1 *_A G_2$, $G_i \hookrightarrow G_1 *_A G_2$ (if $A \hookrightarrow G_i$ injective) are injective. This is not the case when the maps $A \hookrightarrow G_i$ are not supposed injective).